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ON THE CONTINUITY IN $\mathrm{BV}(\Omega)$ OF THE L^2-PROJECTION INTO FINITE ELEMENT SPACES

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ABSTRACT. We show how to obtain continuity in the $\mathrm{BV}(\Omega)$-seminorm of the L^2-projection of $u \in \mathrm{BV}(\Omega)$ into a large class of finite element spaces.

1. INTRODUCTION

In this paper we prove that the L^2-projection (into a large class of finite element function spaces, V_h) of $u \in V = \mathrm{BV}(\Omega)$ is continuous in the BV-seminorm. The set Ω will be taken to be a bounded domain of $\mathbb{R}^n$ with Lipschitz-continuous boundary. The space $\mathrm{BV}(\Omega)$ is defined to be the set of functions of bounded variation in Ω [13]. In this paper we shall take $n = 1, 2, 3$. For some special spaces V_h we can take $n \in \mathbb{N}$.

In the case in which V_h is the space of functions which are constant when restricted to each of the elements of the triangulation $\mathscr{T}_h$ (obtained by a Cartesian product of one-dimensional partitions), the continuity of the L^2-projection in $\mathrm{BV}(\mathbb{R}^n)$ is a well-known result. It has been used by several authors [14, 6, 16, 4] in the error analysis of schemes for numerically solving conservation laws. For more general discontinuous finite element spaces V_h no results seem to be available. The need for this kind of results was prompted by the recent error analysis of monotone schemes defined in general triangulations [5]. In [5] the case in which the finite element space V_h is a space of piecewise constant functions is considered. In this paper a general approach which works for a large class of finite element spaces is presented.

In the two-dimensional case, Crouzeix and Thomée [7] have obtained the continuity of the L^2-projection into V_h for $V = \mathring{W}^{1,p}(\Omega)$ for $1 \leq p \leq \infty$, where $V_h \subset \mathscr{C}^0(\Omega)$ is a standard finite element space of the Lagrangian type. By taking $p = 1$ and using a density argument [13], their results can be trivially proved to hold for $V = \mathring{\mathrm{BV}}(\Omega)$, the space of functions of $\mathrm{BV}(\Omega)$ whose trace is identically zero [13]. This is the single result of the sort available for continuous finite element spaces.

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The technique used in [7], see also [8, 9, and 10], is based on a careful study of the decay of the L^2-projection outside the support of the projected function. Our technique is rather different in nature. It is based on the following three basic ingredients:

(1) definition of the BV-seminorm by duality,

(2) the use of the Π_h-projection introduced in the framework of mixed finite element methods for second-order elliptic problems (see [1, 2], and the bibliography therein), and

(3) the classical approximation results in the theory of finite element methods for second-order elliptic problems [3].

In §2 we state and prove our basic continuity lemmas. They give sufficient conditions which ensure the continuity of the L^2-projection in the $\mathrm{BV}(\Omega)$-seminorm. In §3 we consider the case of triangulations made of simplexes. In §4 we consider the case of a fairly large class of triangulations. Our main results, Theorems 4.1 and 4.2, are obtained as a slight generalization of the corresponding results of §3. We end with some concluding remarks in §5.

2. The basic approximation results

Let Ω be a bounded domain of $\mathbb{R}^n$ which can be expressed as the finite union of n-simplexes. (This implies that the boundary of Ω is Lipschitz continuous.) The total variation of a function $u \in L^1(\Omega)$ is defined to be [13, Definition 1.1],

$$\int_\Omega |Du| = \sup_{\mathbf{w}\in\mathscr{C}_0^1(\Omega;\mathbb{R}^n)} \frac{(u\,,\,\operatorname{div}\mathbf{w})}{\|\mathbf{w}\|_{L^\infty(\Omega)}}. \tag{2.1}$$

Thus, the space $\mathrm{BV}(\Omega)$ is the space of functions $u \in L^1(\Omega)$ with bounded variation in Ω, i.e., such that $\int_\Omega |Du| < \infty$. When Ω has a Lipschitz-continuous boundary, the trace operator, γ, is well defined over $\mathrm{BV}(\Omega)$ [13, Theorem 2.10]. We shall need the following density result.

Lemma 2.1 (Density of $\mathscr{C}^\infty(\Omega)$ in $\mathrm{BV}(\Omega)$). *Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Then, for every $u \in \mathrm{BV}(\Omega)$ there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathscr{C}^\infty(\Omega)$ such that*

(1) $\gamma(u_j) = \gamma(u)\ \forall j \in \mathbb{N}$,

(2) $\| u - u_j \|_{L^1(\Omega)} \to 0$ *as* $j \to \infty$,

(3) $\| \mathbf{grad}\, u_j \|_{L^1(\Omega)} \to \int_\Omega |Du|$ *as* $j \to \infty$.

These results follow from [13, Remark 2.12].

We now make our main assumption: there exists a finite-dimensional space $\mathscr{V}_h \subset H(\operatorname{div};\Omega)$, and a projection $\Pi_h\colon \mathscr{C}^1(\Omega) \to \mathscr{V}_h$ such that

$$(u_h\,,\,\operatorname{div}\mathbf{w}) = (u_h\,,\,\operatorname{div}\Pi_h\mathbf{w}) \qquad \forall\, u_h \in V_h\,, \tag{2.2a}$$

$$\operatorname{trace}_e(\Pi_h\mathbf{w})\cdot\mathbf{n}_{\partial\Omega} = 0 \qquad \forall\,\mathbf{w} \in \mathscr{C}_0^1(\Omega;\mathbb{R}^n)\,, \tag{2.2b}$$

$$\|\Pi_h\mathbf{w}\|_{L^\infty(\Omega)} \le C_1\|\mathbf{w}\|_{L^\infty(\Omega)} \qquad \forall\,\mathbf{w} \in \mathscr{C}^1(\Omega;\mathbb{R}^n)\,, \tag{2.2c}$$

where $\mathbf{n}_{\partial\Omega}$ is the outward unit normal to $\partial\Omega$, and

$$\operatorname{div} \mathscr{V}_h \subset V_h. \tag{2.3}$$

Spaces $\mathscr{V}_h$ and operators Π_h satisfying these properties have been introduced in the framework of the analysis of mixed methods for second-order elliptic equations, see [1, 2] and the bibliography therein. Our first result is the following.

Lemma 2.2 (First Basic Continuity Lemma). *Suppose there is an operator* Π_h *satisfying the conditions* (2.2) *and* (2.3). *Then, for* $u \in \mathrm{BV}(\Omega)$,

$$\int_\Omega |D\mathbb{P}_{V_h} u| \le C_1 \int_\Omega |Du|,$$

where $\mathbb{P}_{V_h} : L^1(\Omega) \to V_h$ *is the* L^2*-projection into* V_h.

Condition (2.3) strongly restricts the class of spaces V_h. Indeed, in [1] and [2] the space V_h is always a space of discontinuous functions. To consider spaces V_h included in $\mathscr{C}^0(\Omega)$ we proceed as follows. Let us denote by $\mathbb{T}_h$ a triangulation of Ω of which $\mathscr{T}_h$ is a refinement. We ask that the following inverse estimate be satisfied:

$$\|\operatorname{div} \Pi_h \mathbf{w}\|_{L^\infty(T)} \le C_2 \hat{h}_T^{-1} \|\Pi_h \mathbf{w}\|_{L^\infty(T)} \qquad \forall \mathbf{w} \in \mathscr{C}^1(T) \qquad \forall T \in \mathbb{T}_h, \tag{2.4}$$

and that the finite element space V_h satisfy the following approximation property:

$$\|u - \mathbb{P}_{V_h} u\|_{L^1(T)} \le C_3 \hat{h}_T \|\mathbf{grad}\, u\|_{L^1(T)}, \qquad u \in \mathscr{C}^\infty(T) \ \ \forall T \in \mathbb{T}_h, \tag{2.5}$$

where $\hat{h}_T = \max\{\operatorname{diam} K : T \supset K \in \mathscr{T}_h\}$. We have the following result.

Lemma 2.3 (Second Basic Continuity Lemma). *Suppose that there is an operator* Π_h *and a finite element space* V_h *satisfying* (2.2), (2.4), *and* (2.5). *Then, for* $u \in \mathrm{BV}(\Omega)$,

$$\int_\Omega |D\mathbb{P}_{V_h} u| \le (C_1 + C_2 C_3) \int_\Omega |Du|.$$

Notice that the only requirement on the space V_h is that it satisfies the approximation property (2.5). A large class of finite element spaces satisfy such a property [3, p. 111 and §3.1]. It is important to point out that the continuity of the L^2-projection $\mathbb{P}_{V_h}$ from $L^1(T)$ to $L^1(T)$ is implicitly required for (2.5) to hold. In [10] such a property has been obtained under the assumption of quasi-uniformity of the triangulation $\{K \in \mathscr{T}_h : K \subset T\}$. Using this assumption, the global inverse estimate (2.4) follows easily from the usual local inverse estimates (obtained by a classical scaling argument [3, Theorem 3.1.2]).

Proof of Lemmas 2.2 *and* 2.3. First, let us prove Lemma 2.3. Pick an element of $\mathrm{BV}(\Omega)$, say u. By Lemma 2.1 there is a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathscr{C}^\infty(\Omega)$ converging to u strongly in $L^1(\Omega)$. Since V_h is a finite-dimensional space, the sequence

$\{\mathbb{P}_{V_h} u_j\}_{j\in\mathbb{N}}$ converges to $\mathbb{P}_{V_h} u$ strongly in $L^1(\Omega)$. Thus, by semicontinuity [13, Theorem 1.9]

$$\int_\Omega |D\,\mathbb{P}_{V_h} u| \leq \liminf_{j\to\infty} \int_\Omega |D\,\mathbb{P}_{V_h} u_j|.$$

But, by (2.2a) and (2.2b),

$$\begin{aligned}(\mathbb{P}_{V_h} u_j , \operatorname{div} \mathbf{w}) &= (\mathbb{P}_{V_h} u_j , \operatorname{div} \Pi_h \mathbf{w}) \\ &= (u_j , \operatorname{div} \Pi_h \mathbf{w}) + (\mathbb{P}_{V_h} u_j - u_j , \operatorname{div} \Pi_h \mathbf{w}) \\ &= -(\mathbf{grad}\, u_j , \Pi_h \mathbf{w}) + (\mathbb{P}_{V_h} u_j - u_j , \operatorname{div} \Pi_h \mathbf{w}) \\ &= -(\mathbf{grad}\, u_j , \Pi_h \mathbf{w}) + \sum_{T\in\mathbb{T}_h} (\mathbb{P}_{V_h} u_j - u_j , \operatorname{div} \Pi_h \mathbf{w})_T ,\end{aligned}$$

and by (2.2c), (2.4) and (2.5),

$$\int_\Omega |D\,\mathbb{P}_{V_h} u_j| \leq (C_1 + C_2 C_3)\, \|\mathbf{grad}\, u_j\|_{L^1(\Omega)}.$$

Lemma 2.3 follows by combining the two above inequalities and using (3) of Lemma 2.1.

Now, let us prove Lemma 2.2. We proceed as above, and notice that if condition (2.3) is satisfied,

$$(\mathbb{P}_{V_h} u_j - u_j , \operatorname{div} \Pi_h \mathbf{w}) = 0.$$

Hence, in this case the hypotheses (2.4) and (2.5) are superfluous. This proves Lemma 2.2. □

3. Triangulations made of simplexes

Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^n$. In this section and in §4 we require that for every $K_1, K_2 \in \mathcal{T}_h$

(3.1) if the $(n-1)$-dimensional Lebesgue measure of $S = K_1 \cap K_2$ is not 0, then S is a full $(n-1)$-dimensional face of both ∂K_1 and ∂K_2.

Set

$$\sigma_h = \max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K},$$

where $h_K = \operatorname{diam} K$, and $\rho_K = \sup\{\rho\colon$ a ball of diameter ρ is included in $K\}$. If σ_h is uniformly bounded, then the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is called regular [3, p. 132]. Many of the constants C that appear in our continuity results do depend on σ_h. It is implicitly assumed that if σ_h is uniformly bounded, i.e., if the family $\{\mathcal{T}_h\}_{h>0}$ is regular, the constant C can be taken to be independent of h. In this section we shall only consider triangulations made of simplexes.

Theorem 3.1. *Let V_h be the space of functions whose restriction to each $K \in \mathcal{T}_h$ is constant. Then, for $u \in \mathrm{BV}(\Omega)$,*

$$\int_\Omega |D\mathbb{P}_{V_h}u| \le C\int_\Omega |Du|,$$

where $C = n^{3/2}\sigma_h$.

The dependence of the constant C on σ_h is sharp. To see this, consider the two-dimensional case and take u to be the characteristic function of $[0, 1]^2$. Take, for example, $\Omega = (-0.5, 1.5)^2$. It is well known [13, Example 1.4] that

$$\int_\Omega |Du| = 4.$$

Consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ made of triangles, see Figures 1 and 2. Let V_h be the space of functions which are constant when restricted to

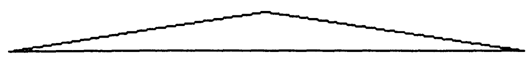

FIGURE 1
A typical triangle of triangulation $\mathcal{T}_h$. The size of the biggest side is h, and the height is $(1-h)/(2n)$. Note that for this triangle $\sigma_h = \nu + \sqrt{\nu^2+1}$ where $\nu = nh/(1-h)$.

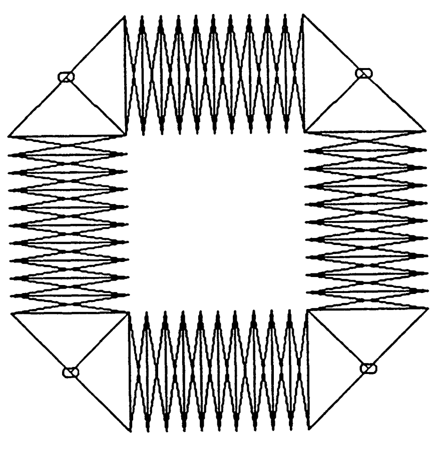

FIGURE 2
The relevant triangles of a typical $\mathcal{T}_h$. The '0' represent the vertices of the square $[0, 1]^2$. The function $\mathbb{P}_{V_h}u$ is equal to $1/2$ on the triangles displayed. It is equal to 1 inside the remaining of $[0, 1]^2$, and equal to 0 elsewhere. In this case $h = 0.4$ and $n = 10$.

each element $K \in \mathcal{T}_h$. A straightforward computation shows that

$$\int_\Omega |D\mathbb{P}_{V_h} u| = f(h, \sigma_h)\sigma_h \int_\Omega |Du|,$$

where

$$f(h, \sigma_h) = \frac{h}{\sqrt{2}\,\sigma_h} + \frac{1}{2}(1-h)\left(1 + \frac{1}{\sigma_h^2}\right).$$

Theorem 3.1 is a particular case of the next result; see also [5]. We shall denote by $P^{k-1}(K)$ the space of polynomials (defined on the interior of the set K) of total degree equal or smaller than $k-1$.

Theorem 3.2. *Let n be either* 1, 2, *or* 3. *Let V_h be the finite element space whose elements belong to $P^{k-1}(K)$ when restricted to the elements $K \in \mathcal{T}_h$, for some integer $k \geq 1$. Then, there exists a constant $C > 0$ such that for $u \in \mathrm{BV}(\Omega)$*

$$\int_\Omega |D\mathbb{P}_h u| \leq C \int_\Omega |Du|.$$

The constant C depends solely on k, n and σ_h.

Note that V_h is a space of discontinuous functions. We shall prove the result for $n = 3$. The other cases are proven in an analogous way.

Following Brezzi et al. [2, equations 2.4], we define the operator Π_h on $\mathscr{C}_0^1(\Omega)$ as follows:

$$(\Pi_h \mathbf{w})|_K = \Pi_K(\mathbf{w}|_K) \qquad \forall K \in \mathcal{T}_h,$$

where $\Pi_K : \mathscr{C}^1(K) \to \mathbf{P}^k(K)$ ($\mathbf{P}^k(K)$ is the vector analogue of $P^k(K)$ consisting of three copies of $P^k(K)$) is defined by the following relations:

(3.2a) $$\langle (\mathbf{w} - \Pi_K \mathbf{w}) \cdot \mathbf{n}_{e,K}, p \rangle_e = 0 \qquad \forall p \in P^k(e),\, e \in \partial K,$$

(3.2b) $$(\mathbf{w} - \Pi_K \mathbf{w}, \mathbf{grad}\, w)_K = 0 \qquad \forall w \in P^{k-1}(K),$$

(3.2c) $$(\mathbf{w} - \Pi_K \mathbf{w}, \mathbf{v})_K = 0 \qquad \forall \mathbf{v} \in \mathbf{B}^k(K),$$

where

(3.3) $$\mathbf{B}^k(K) = \{\mathbf{u} \in \mathbf{P}^k(K) : \mathbf{u} \cdot \mathbf{n}_{e,K} = 0 \quad \forall e \in \partial K, \quad \text{and } (\mathbf{u}, \mathbf{grad}\, w) = 0 \quad \forall w \in P^{k-1}(K)\}.$$

Lemma 3.3 (Brezzi et al. [2]). *The projection Π_h defined by* (3.2), (3.3) *satisfies conditions* (2.2a) *and* (2.3).

Lemma 3.4. *The projection Π_h defined by* (3.2), (3.3) *satisfies the condition* (2.2b).

Proof. If $e \in \partial\Omega$, then $\langle \Pi_K \mathbf{w} \cdot \mathbf{n}_{e,K}, p \rangle_e = 0$, $\forall\, p \in P^k(e)$, by (3.2a). Since $\mathrm{trace}_e(\Pi_K \mathbf{w}) \cdot \mathbf{n}_{e,K}$ belongs to $P^k(e)$, the property (2.2b) follows. □

Lemma 3.5. *The projection Π_h defined by* (3.2), (3.3) *satisfies the condition* (2.2c) *with a constant C_1 which depends solely on k, n, and σ_h.*

The proof is obtained by classical scaling arguments [3, Theorem 3.2.1] and the properties (3.2).

Proof of Theorem 3.2. The assertion follows immediately from Lemmas 3.3, 3.4, 3.5, and 2.2. $\square$

We now extend Theorem 3.2 to the case in which V_h is a general finite element space. Set

$$\kappa_{h,T} = \max_{T\supset K\in\mathcal{T}_h} \frac{\hat{h}_T}{h_K}, \qquad \kappa_h = \max_{T\in\mathbb{T}_h} \kappa_{h,T}.$$

Recall that $\hat{h}_T = \max\{h_K : T \supset K \in \mathcal{T}_h\}$. Notice that κ_h depends on both triangulations $\mathcal{T}_h$ and $\mathbb{T}_h$. Let us consider the families of triangulations $\{\mathcal{T}_h\}_{h>0}$ and $\{\mathbb{T}_h\}_{h>0}$. If κ_h is uniformly bounded, we say that $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform with respect to $\{\mathbb{T}_h\}_{h>0}$. Let us justify this terminology. If $\mathbb{T}_h = \{\Omega\}$, and if κ_h is uniformly bounded, it is customary to say that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. On the other hand, if $\mathbb{T}_h = \mathcal{T}_h$, then $\kappa_{h,T} \equiv 1$ and $\kappa_h \equiv 1$ for all $h > 0$; in other words, $\mathcal{T}_h$ is always quasi-uniform with respect to itself. The interesting case is when $\mathbb{T}_h$ is neither $\{\Omega\}$ nor $\mathcal{T}_h$; in this case the boundedness of σ_h is an indication of a sort of 'local quasi-uniformity' of the family $\{\mathcal{T}_h\}_{h>0}$. It is implicitly assumed that if κ_h is uniformly bounded, i.e., if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform with respect to $\{\mathbb{T}_h\}_{h>0}$, the constants C appearing in some of our continuity results can be taken to be independent of h.

Theorem 3.6. *Let* n *be either* 1, 2, *or* 3. *Let* V_h *be a finite element space whose functions belong to* $P^{k-1}(K)$ *when restricted to* $K \in \mathcal{T}_h$, *for some integer* $k \geq 1$. *Suppose that* V_h *satisfies the approximation property* (2.5). *Then there is a constant* $C > 0$ *such that for* $u \in \mathrm{BV}(\Omega)$,

$$\int_\Omega |D\mathbb{P}_{V_h} u| \leq C \int_\Omega |Du|.$$

The constant C *depends solely on* n, k, κ_h, σ_h, *and* C_3.

Proof. We proceed as in the proof of Theorem 3.2 to show the existence of a projection Π_h satisfying (2.2). By a classical scaling argument,

$$\|\operatorname{div} \Pi_h \mathbf{w}\|_{L^\infty(K)} \leq C h_K^{-1} \|\Pi_h \mathbf{w}\|_{\mathbf{L}^\infty(K)} \qquad \forall K \in \mathcal{T}_h,$$

where C depends solely on σ_h, k, and n. Property (2.4) is thus satisfied with $C_2 = C\kappa_h$. Finally, the result follows from a direct application of Lemma 2.3. $\square$

4. General triangulations

The results of §3 can be proven to hold for triangulations made of rectangles by using the finite element methods of Brezzi et al.; see [1, Lemma 5.1] for the case $n = 2$, and [2, §3] for the case $n = 3$. In this important case it can easily be proven that the continuity constant C does not depend on the regularity of the hypercubes, i.e., on the quantity σ_h. This is in sharp contrast with the simplexes case in which the continuity constant blows up when the simplexes become flatter and flatter.

The case of general quadrangles can be considered by using the finite elements of Girault and Raviart [12]. Also, triangulations made of prisms can be handled by using the elements introduced by Nédélec [15, §2.3]. We now show a unified approach devised to handle more general triangulations.

Let $\mathcal{T}_h$ be any triangulation of Ω for which there exists a refinement $\widetilde{\mathcal{T}_h}$ (made only of simplexes) satisfying (3.1) (notice that the triangulation $\mathcal{T}_h$ need not satisfy condition (3.1)). In other words, $\mathcal{T}_h$ is any triangulation whose elements are a union of simplexes. We set

$$\tilde{\sigma}_h = \max_{\widetilde{K}\in\widetilde{\mathcal{T}_h}} \frac{h_{\widetilde{K}}}{\rho_{\widetilde{K}}},$$

$$\forall K \in \mathcal{T}_h: \qquad \tilde{\kappa}_{h,K} = \max_{K\supset\widetilde{K}\in\widetilde{\mathcal{T}_h}} \frac{\tilde{h}_K}{h_{\widetilde{K}}},$$

$$\tilde{\kappa}_h = \max_{K\in\mathcal{T}_h} \tilde{\kappa}_{h,K},$$

where $\tilde{h}_K = \max\{h_{\widetilde{K}} : K \supset \widetilde{K} \in \widetilde{\mathcal{T}_h}\}$. Notice that the refinement $\widetilde{\mathcal{T}_h}$ is not unique, and hence the quantities $\tilde{\sigma}_h$ and $\tilde{\kappa}_h$ could be rendered smaller by a suitable choice of $\widetilde{\mathcal{T}_h}$. The following result is a generalization of Theorems 3.1 and 3.2.

Theorem 4.1. *Let n be either* 1, 2, *or* 3. *Let V_h be the finite element space of functions which belong to $P^{k-1}(K)$ when restricted to $K \in \mathcal{T}_h$, for some $k \geq 1$ (if $k = 1$, then n can be an arbitrary natural number). Then there exists a constant C such that, for $u \in \mathrm{BV}(\Omega)$,*

$$\int_\Omega |D\mathbb{P}_{V_h} u| \leq C \int_\Omega |Du|,$$

where C depends on n, k, $\tilde{\sigma}_h$, and $\tilde{\kappa}_h$.

Proof. First, notice that since $u \in V_h$, its restriction to $K \in \widetilde{\mathcal{T}_h}$ belongs to $P^{k-1}(K)$. Thus, $V_h \subset \widetilde{V}_h$, where $\widetilde{V}_h$ is the set of functions whose restriction to $K \in \widetilde{\mathcal{T}_h}$ belongs to $P^{k-1}(K)$. Proceeding as in §3, we easily see that the conditions (2.2) are satisfied (notice that (2.2a) holds for every $u_h \in \widetilde{V}_h \supset V_h$!).

The condition (2.4) is satisfied with $\mathbb{T}_h = \widetilde{\mathcal{T}_h}$ and a constant C_2 which depends on $\tilde{\sigma}_h$. Moreover, since

$$C_2 h_{\widetilde{K}}^{-1} \leq C_2 \tilde{\kappa}_h h_K^{-1},$$

condition (2.4) is also satisfied with $\mathbb{T}_h = \mathcal{T}_h$, with the "new" C_2 depending on $\tilde{\kappa}_h$ and $\tilde{\sigma}_h$.

To obtain condition (2.5), we proceed as follows. From the continuity of the (restriction to each element $T \in \mathbb{T}_h$) L^2-projection in L^1 and classical interpolation results [3, Theorem 3.1.4], we get

$$\| u - \mathbb{P}_{V_h} u \|_{L^1(K)} \leq C_K h_K \| \mathbf{grad}\, u \|_{L^1(K)}, \qquad \forall K \in \mathscr{T}_h,$$

where the constant C_K depends on the shape of the element K. If all the elements are affinely equivalent to a single reference element, all the C_K can be taken to be equal to each other. In this case, $C_3 = \sup_{K \in \mathscr{T}_h} C_K$ is thus independent of h. This fact remains true if the number of reference elements is finite. However, since the elements K are constructed as general unions of simplexes, the number of reference elements could grow unboundedly as h goes to zero.

To deal with this case, we can use the approximation results obtained in [11]. We shall only outline how to use such results: First, we write the projection $P_{V_h}|_K$ as a convex combination of local projections defined on the restriction of u to open sets which include only one or two adjacent elements $\tilde{K}$. Second, we apply [11, Theorem 3.2] to each of those local projections. Third, we apply [11, Theorem 7.1] with our local projections playing the role of the operators $\mathscr{Q}_j$ in [11]. (Notice that K is connected, and that the number of elements $\tilde{K}$ is finite (and depending on $\tilde{\sigma}_h$ and $\tilde{\kappa}_h$).) Finally, we use the triangle inequality to obtain the estimate for the initial projection. In this way the constant C_3 can be proven to depend solely on n, k, $\tilde{\sigma}_h$, and $\tilde{\kappa}_h$.

We can now apply Lemma 2.3 to obtain our result. □

Our final result generalizes Theorem 3.6. Its proof is similar to the proof of Theorem 4.1.

Theorem 4.2. *Let n be either 1, 2, or 3. Let V_h be a finite element space of functions which belong to $P^{k-1}(K)$ when restricted to $K \in \mathscr{T}_h$, for some $k \geq 1$ (if $k = 1$, then n can be an arbitrary natural number). Suppose that V_h satisfies the approximation property (2.5). Then there exists a constant C such that, for $u \in \mathrm{BV}(\Omega)$,*

$$\int_\Omega |D \mathbb{P}_{V_h} u| \leq C \int_\Omega |D u|,$$

where C depends on n, k, $\tilde{\sigma}_h$, $\tilde{\kappa}_h$ and C_3.

We end this section by pointing out that all the results of this paper can be extended to the case in which the domain Ω has a curved boundary, provided $u \in \overset{\circ}{\mathrm{BV}}(\Omega)$; see [1, equations (2.4)] for the use of triangles with one curved edge, [1, equations (5.5)] for rectangles with one curved edge, [2, equations (2.2)] for tetrahedra with one curved face, and [2, equations (3.6)] for cubes with one curved face. The reason for taking $u \in \overset{\circ}{\mathrm{BV}}(\Omega)$ is as follows. When Ω has a curved boundary, the condition (2.2b) does not hold for the (known) operators Π_h. Notice that if $e \in \partial\Omega$, then the function $\mathrm{trace}_e(\Pi_h \mathbf{w}) \cdot \mathbf{n}_{K,e}$ is not a polynomial in general; moreover, the argument in the proof of Lemma 3.4 cannot be used because the definition of the projections Π_h in [1], [2] is different from the definition (3.2) in that (3.2a) holds now only for $e \in \partial K \setminus \partial\Omega$. The missing condition is replaced by a condition involving values of $\mathbf{w}$ in the interior of K. If $u \in \overset{\circ}{\mathrm{BV}}(\Omega)$, i.e., if $\gamma(u) = 0$, the condition (2.2b) is unnecessary for our Basic Continuity Lemmas to hold.

5. Concluding remarks

In this paper we have shown how to obtain the continuity of the L^2-projection (into a large class of finite element spaces) in the $\mathrm{BV}(\Omega)$-seminorm of functions $u \in \mathrm{BV}(\Omega)$. We use a new technique based on duality. It involves the use of the Π_h-projection used in the framework of mixed methods for second-order elliptic problems [1], [2], [15]. Our basic results are Lemmas 2.2 and 2.3. Our main results are Theorems 4.1 and 4.2. This continuity result has already been used (with $k = 1$) in the analysis of the convergence of monotone schemes (defined in general triangulations) for scalar conservation laws [5]. We believe that the results of this paper will be useful in the framework of the error analysis in $L^\infty(L^1)$ of discontinuous or continuous finite element methods for conservation laws.

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